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Theory of acoustic eigenmodes in parabolic cylindrical enclosures

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Abstract

We present an exact method for calculating eigenmodes and eigenfrequencies for an acoustic enclosure (AE) defined by two confocal parabolic cylinders and two plane surfaces using parabolic cylinder coordinates. Rigid-wall boundary conditions are assumed; however, the model can easily be extended to include the more general case with mixed boundary conditions. A discussion of eigenmode symmetry properties and mode frequency dependence on AE shape is given. The latter problem is addressed by considering a series of symmetrical and asymmetrical AEs having the same volume.
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1. Introduction

The Helmholtz equation is known to be separable in 11 coordinate systems including the parabolic cylinder coordinates (PCC) [1,2]. In this paper, an exact method is applied to determine eigenmodes and eigenfrequencies in an acoustic enclosure (AE) defined by two confocal parabolic cylinders and two plane surfaces subject to rigid-wall boundary conditions ($\nabla P \cdot \mathbf{n} = 0$, where P is the acoustic pressure and $\mathbf{n}$ is a surface normal vector). The medium confined by the AE is assumed to be an ideal fluid with spatially uniform properties. The problem is solved

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quasi-analytically using the Frobenius power series expansion technique based on the separability of the eigenfunction. In contrast to numerical methods such as the finite-element method (FEM) and the boundary-element method (BEM) [3–6], the present analytical method allows eigenmodes and eigenfrequencies to be obtained exactly including the full parameter dependencies and symmetry properties. The corresponding cases with AEs of rectangular, spherical, and cylindrical shapes are well-known from textbook examples. The reason for studying the AE problem in PCC is motivated by the need to understand better the sound field in AEs (and waveguides) with lens-shaped cross-sections at intermediate frequencies where both geometrical acoustical and ray-tracing models fail to provide accurate results. There has been previous work on elliptic cylinders [7,8], motivated in part by application to mufflers and ducts. We show here that the parabolic cylinder problem is, in fact, even easier to solve. Since the cross-section is determined by two surfaces, a parabolic cylinder also allows a higher degree of engineering.

In the next section, we show how the Helmholtz equation is separated using PCC and present a series solution of the resulting ordinary differential equations. We then present numerical results for various AEs.

2. Theory

2.1. Helmholtz equation in PCC

In this section, we show how to calculate the eigenmodes and eigenfrequencies for an AE defined by two confocal parabolic cylinders ($|v| = v_0$ and $\mu = \mu_0$) and two plane surfaces $z = z_1$ and $z = z_2$ (Fig. 1). The relation between the PCC and cartesian coordinates is as follows:

$$\begin{aligned} &x = \tfrac{1}{2}(\mu^2 - v^2),\\ &y = \mu v,\\ &z = z,\\ &0 \leqslant \mu < \infty,\ -\infty < v < +\infty,\ -\infty < z < +\infty. \end{aligned} \tag{1}$$

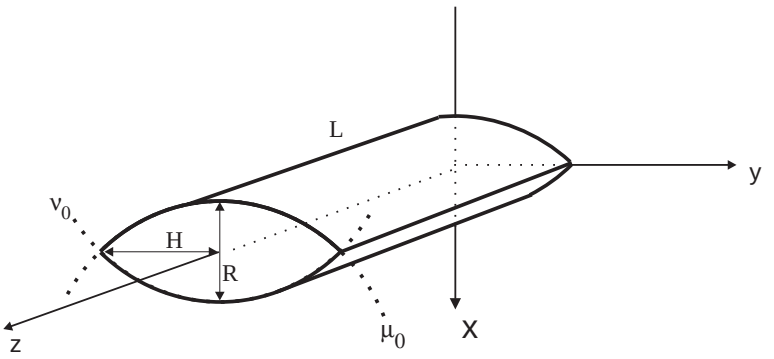

Fig. 1. Geometry of AE (cross-sectional view in z plane).

We have chosen the range of v to include negative values in order to cover the $y<0$ region. The expressions for the various geometrical parameters given in Fig. 1 are as follows:

$$R = x_{\max} - x_{\min} = \tfrac{1}{2}(\mu_0^2 + v_0^2), \tag{2}$$

$$H = \tfrac{1}{2}(y_{\max} - y_{\min}) = \mu_0 v_0, \tag{3}$$

$$V = \tfrac{2}{3} L\mu_0 v_0(\mu_0^2 + v_0^2) = \tfrac{4}{3} LRH, \tag{4}$$

where L is the axial length of the AE (along the z-axis), R is the radius (along the x-axis), H is the height (along the y-axis), and V is the AE volume. The volume is obtained by integrating a volume element in PCC:

$$V = \int_{z_1}^{z_2} \int_0^{\mu_0} \int_{-v_0}^{v_0} (\mu^2 + v^2)\, \mathrm{d}z\, \mathrm{d}\mu\, \mathrm{d}v \tag{5}$$

and use is made of $L = z_2 - z_1$.

The eigenmodes P and eigenfrequencies f of an AE are obtained by solving the Helmholtz equation

$$\nabla^2 P(\mathbf{r}) + k^2 P(\mathbf{r}) = 0, \tag{6}$$

subject to the rigid-wall boundary condition

$$\nabla P \cdot \mathbf{n}|_{(\mu=\mu_0, |v|=v_0, z=z_1, z=z_2)} = 0, \tag{7}$$

where $k = \omega/c$, ω is the angular frequency ($= 2\pi f$), and c is the speed of sound in the medium inside the AE. The Laplacian in PCC reads:

$$\nabla^2 P = \frac{1}{\mu^2 + v^2}\left[\frac{\partial^2 P}{\partial \mu^2} + \frac{\partial^2 P}{\partial v^2}\right] + \frac{\partial^2 P}{\partial z^2}. \tag{8}$$

Helmholtz's equation is known to be separable in PCC [1]. Writing

$$P(\mu, v, z) = M(\mu)N(v)Z(z), \tag{9}$$

one gets the following separated ordinary differential equations:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\mu^2} - (\alpha + \beta\mu^2)M = 0, \tag{10}$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}v^2} + (\alpha - \beta v^2)N = 0, \tag{11}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + (k^2 + \beta)Z = 0, \tag{12}$$

where α and β are separation constants to be determined in the following.

Series solutions to the ordinary differential equations (10)–(12) can be found using the Frobenius method [2]. Since there are no singular points in Eqs. (10)–(12), two independent and well-behaved solutions exist for each separated equation. Consider the differential equation in M first. Expanding M about $\mu = 0$ one writes:

$$M(\mu) = \sum_{n=0}^{\infty} a_n \mu^{n+\kappa}, \tag{13}$$

where κ is a constant to be specified as follows. Inserting Eq. (13) in Eq. (10) and equating terms proportional to $\mu^{\kappa-2}$ and $\mu^{\kappa-1}$ gives two equations:

$$\kappa(\kappa-1)a_0 = 0, \tag{14}$$

$$\kappa(\kappa+1)a_1 = 0. \tag{15}$$

Thus, $\kappa = 0$ or 1 if we choose $a_0 = 1$ and $a_1 = 0$. The two choices for κ will give us two independent series solutions as we will prove next. Applying the identity theorem for power series to terms $\mu^{\kappa+n}$ where $n = 0, 1, 2, \ldots$ give the recursion formula:

$$a_2 = \frac{\alpha}{2}, \qquad a_3 = 0,$$

$$a_{n+4} = \frac{\alpha a_{n+2} + \beta a_n}{(n+4)(n+3)} \quad \text{where } n = 0, 1, 2, 3, \ldots \tag{16}$$

if $\kappa = 0$. This solution for $M(\mu) \equiv M_1(\mu)$ is clearly even in μ.

Similarly, for the second solution for M corresponding to $\kappa = 1$, the following recursion formula is obtained:

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = \frac{\alpha}{6}, \qquad b_3 = 0,$$

$$b_{n+4} = \frac{\alpha b_{n+2} + \beta b_n}{(n+5)(n+4)} \quad \text{where } n = 0, 1, 2, 3, \ldots \tag{17}$$

and the associated solution for $M(\mu) \equiv M_2(\mu)$ is an odd function of μ. The general solution to Eq. (10) is accordingly

$$M(\mu) = M(\alpha, \beta; \mu) = AM_1(\mu) + BM_2(\mu), \tag{18}$$

where A and B are arbitrary constants, and

$$M_1(\mu) = M_1(\alpha, \beta; \mu) = \sum_{n=0}^{\infty} a_{2n}\mu^{2n}, \tag{19}$$

$$M_2(\mu) = M_2(\alpha, \beta; \mu) = \sum_{n=0}^{\infty} b_{2n}\mu^{2n+1}. \tag{20}$$

Consider next the differential equation in N (Eq. (11)). This differential equation is analogous to the differential equation in M (Eq. (10)) except that α must be replaced by $-\alpha$. We obtain

$$N(v) = N(\alpha, \beta; \mu) = CN_1(v) + DN_2(v), \tag{21}$$

where C and D are arbitrary constants, and

$$N_1(v) = N_1(\alpha, \beta; \mu) = \sum_{n=0}^{\infty} c_{2n}v^{2n}, \tag{22}$$

$$N_2(v) = N_2(\alpha, \beta; \mu) = \sum_{n=0}^{\infty} d_{2n}v^{2n+1}. \tag{23}$$

The coefficients c_n satisfy the recurrence relations

$$c_0 = 1, \qquad c_2 = \frac{-\alpha}{2},$$
$$c_{n+4} = \frac{-\alpha c_{n+2} + \beta c_n}{(n+4)(n+3)} \quad \text{where } n = 0, 1, 2, 3, \ldots . \tag{24}$$

For d_n we obtain

$$d_0 = 1, \qquad d_2 = \frac{-\alpha}{6},$$
$$d_{n+4} = \frac{-\alpha d_{n+2} + \beta d_n}{(n+5)(n+4)} \quad \text{where } n = 0, 1, 2, 3, \ldots . \tag{25}$$

The remaining differential equation in Z can be solved immediately so as to give the general solution

$$Z(z) = E \sin\left(\left[\sqrt{k^2+\beta}\right] z\right) + F \cos\left(\left[\sqrt{k^2+\beta}\right] z\right), \tag{26}$$

where E and F are arbitrary constants.

The rigid-wall boundary condition (Eq. (7)) now becomes:

$$\frac{\mathrm{d}M}{\mathrm{d}\mu}(\alpha, \beta; \mu_0) = 0, \tag{27}$$
$$\frac{\mathrm{d}N}{\mathrm{d}v}(\alpha, \beta; v_0) = \frac{\mathrm{d}N}{\mathrm{d}v}(\alpha, \beta; -v_0) = 0, \tag{28}$$
$$\frac{\mathrm{d}Z}{\mathrm{d}z}(k, \beta; z_1) = \frac{\mathrm{d}Z}{\mathrm{d}z}(k, \beta; z_2) = 0, \tag{29}$$

where (μ_0, v_0, z_1, z_2) defines the AE boundary. It turns out that Eqs. (27)–(29) together with normalization are not sufficient to solve the eigenvalue problem. This is due to the vanishing of the Jacobian for the transformation to PCC. It was shown by Lebedev [9] that one also requires that ∇P is finite everywhere inside the AE region. Using this result and

$$(\nabla P)^2 = \frac{1}{\mu^2 + v^2}\left[\left(\frac{\partial P}{\partial \mu}\right)^2 + \left(\frac{\partial P}{\partial v}\right)^2\right] + \left(\frac{\partial P}{\partial z}\right)^2 \tag{30}$$

leads to

$$\left[\left(\frac{\partial P}{\partial \mu}\right)^2 + \left(\frac{\partial P}{\partial v}\right)^2\right]\Bigg|_{\mu=v=0} = 0. \tag{31}$$

Eigenstates and associated frequencies f (or k) will be found by imposing the six conditions given by Eqs. (27)–(29) and (31). These conditions clarify that α and β are determined by solving a set of coupled equations instead of a single dispersion relation.

Combining the general solution (Eq. (21)) with Eq. (28) and making use of the fact that N_1 and N_2 are even and odd, respectively, allows us to write

$$\frac{\mathrm{d}N}{\mathrm{d}v}(\alpha, \beta; v_0) + \frac{\mathrm{d}N}{\mathrm{d}v}(\alpha, \beta; -v_0) = 2D\frac{\mathrm{d}N_2}{\mathrm{d}v}(v_0) = 0. \tag{32}$$

This condition can be fulfilled either if

$$D = 0 \tag{33}$$

or

$$\frac{\mathrm{d}N_2}{\mathrm{d}v}(v_0) = 0. \tag{34}$$

However, if $D = 0$, then $C \neq 0$ for P to be non-trivial. Thus, $\mathrm{d}N_1/\mathrm{d}v(v_0) = 0$ so as to satisfy Eq. (28). Instead, if $\mathrm{d}N_2/\mathrm{d}v(v_0) = 0$, we obtain $\mathrm{d}N/\mathrm{d}v(v_0) = C\mathrm{d}N_1/\mathrm{d}v(v_0) = 0$ leaving two possibilities open: $C = 0$ or $\mathrm{d}N_1/\mathrm{d}v(v_0) = 0$.

The condition in Eq. (31) can be restated as

$$B^2C^2 + A^2D^2 = 0 \tag{35}$$

by use of

$$\begin{aligned} &N(0) = C, \\ &M(0) = A, \\ &\left(\frac{\mathrm{d}M}{\mathrm{d}\mu}\right)^2\bigg|_{\mu=0} = B^2, \\ &\left(\frac{\mathrm{d}N}{\mathrm{d}v}\right)^2\bigg|_{v=0} = D^2. \end{aligned} \tag{36}$$

The latter set of relations follow immediately from Eqs. (18) and (21) and the series expansions of $M_1(\mu)$, $M_2(\mu)$, $N_1(v)$, and $N_2(v)$.

Next, let us consider the three possible cases (a) $C \neq 0$, $D = 0$; (b) $D \neq 0$, $C = 0$; and (c) $C \neq 0$, $D \neq 0$; separately.

Case (a) ($C \neq 0$ *and* $D = 0$): Eq. (34) then implies $\mathrm{d}N_1/\mathrm{d}v(v_0) = 0$. The condition that ∇P is finite at $\mu = v = 0$ now gives $B = 0$ (refer to Eq. (35)). If $B = 0$ then $A \neq 0$ and $\mathrm{d}M_1/\mathrm{d}\mu(\mu_0) = 0$ so as to satisfy Eq. (27). In conclusion, case (a) demands

$$\begin{aligned} &\frac{\mathrm{d}N_1}{\mathrm{d}v}(v_0) = 0, \\ &\frac{\mathrm{d}M_1}{\mathrm{d}\mu}(\mu_0) = 0. \end{aligned} \tag{37}$$

Case (b) ($D \neq 0$ *and* $C = 0$): Thus $\mathrm{d}N_2/\mathrm{d}v(v_0) = 0$ (refer to the discussion following Eqs. (33) and (34)). The condition that ∇P is finite at $\mu = v = 0$ now gives $A = 0$ (refer to Eq. (35)). If $A = 0$ then $B \neq 0$ and $\mathrm{d}M_2/\mathrm{d}\mu(\mu_0) = 0$ so as to satisfy Eq. (27). In conclusion, case (b) demands

$$\begin{aligned} &\frac{\mathrm{d}N_2}{\mathrm{d}v}(v_0) = 0, \\ &\frac{\mathrm{d}M_2}{\mathrm{d}\mu}(\mu_0) = 0. \end{aligned} \tag{38}$$

Case (*c*): It follows immediately from the discussion following Eqs. (33) and (34) that when $C \neq 0$, $D \neq 0$, the two conditions

$$\frac{\mathrm{d}N_1}{\mathrm{d}v}(v_0) = 0,$$
$$\frac{\mathrm{d}N_2}{\mathrm{d}v}(v_0) = 0, \tag{39}$$

must be imposed. If $\mathrm{d}M_1/\mathrm{d}\mu(\mu_0) = 0$ by chance then $\mathrm{d}M_2/\mathrm{d}\mu(\mu_0) = 0$ so as to satisfy Eqs. (27) and (35) (Eq. (35) forces $B = 0$ if $A = 0$, $C \neq 0$, and $D \neq 0$, which is not possible for a non-trivial solution). Similarly, if $\mathrm{d}M_2/\mathrm{d}\mu(\mu_0) = 0$ by accidence then $\mathrm{d}M_1/\mathrm{d}\mu(\mu_0) = 0$. These two special cases therefore require *four* conditions to be fulfilled: $\mathrm{d}M_1/\mathrm{d}\mu(\mu_0) = \mathrm{d}M_2/\mathrm{d}\mu(\mu_0) = \mathrm{d}N_1/\mathrm{d}v(v_0) = \mathrm{d}N_2/\mathrm{d}v(v_0) = 0$. If instead both $\mathrm{d}M_1/\mathrm{d}\mu(\mu_0) \neq 0$ and $\mathrm{d}M_2/\mathrm{d}\mu(\mu_0) \neq 0$, it is always possible to find non-zero (complex) coefficients A and B such that Eqs. (27) and (31) are satisfied simultaneously. In the latter case, the two conditions given by Eq. (39) are the *only* conditions that must be satisfied for an eigenstate to be found.

2.2. Relation between k and β

Consider next the Z equation (Eq. (26)) and the associated boundary conditions (Eq. (29)). Two possible cases can occur if we choose the origin of the z-axis such that $z_2 = -z_1 = L/2$: (i) $E = 0$ or (ii) $F = 0$. If $E = 0$, then the following relation between k and β must hold:

$$k^2 = (2p)^2\left(\frac{\pi}{L}\right)^2 - \beta \quad \text{where } p = 0, 1, 2, 3, \ldots . \tag{40}$$

If, instead, $F = 0$, we find

$$k^2 = -\beta \quad \text{or}$$
$$k^2 = (2p+1)^2\left(\frac{\pi}{L}\right)^2 - \beta \quad \text{where } p = 0, 1, 2, 3, \ldots . \tag{41}$$

Eqs. (41) and (40) can be restated in terms of the eigenfrequency as follows:

$$f = \frac{c}{2\pi}\sqrt{(2p)^2\left(\frac{\pi}{L}\right)^2 - \beta} \quad \text{where } p = 1, 2, 3, \ldots \tag{42}$$

if $E = 0$, and

$$f = \frac{c}{2\pi}\sqrt{-\beta} \quad \text{or}$$
$$f = \frac{c}{2\pi}\sqrt{(2p+1)^2\left(\frac{\pi}{L}\right)^2 - \beta} \quad \text{where } p = 0, 1, 2, 3, \ldots \tag{43}$$

if $F = 0$. The procedure for determining eigenstates is as follows. For each case (a), (b), and (c), solution sets (α, β) are found by solving Eqs. (37), (38), and (39), respectively. The values found for β can finally be inserted in Eqs. (43) and (42) to get f.

Let us also briefly comment on the case where the parabolic cylinder enclosure is excited by an external force with known frequency. Eigenvalues are then found by using the following

procedure. Consider case (a) for which $B = D = E = 0$, a typical eigenmode is:

$$\psi_{pn}(\mu, \nu, z) = M_1(\alpha_{pn}, \beta_p, \mu) N_1(\alpha_{pn}, \beta_p, \nu) \cos(2p\pi z/L), \tag{44}$$

where $\beta_p = (2p\pi/L)^2 - k^2$, $p = 0, 1, 2, \ldots$, and k is known. Under such circumstances what is of interest is the eigenvalues α_{pn}, $p = 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$. These can be found by first taking $p = 0$ and solving Eq. (37). This procedure can then be repeated for $p = 1, 2, 3, \ldots$.

3. Results and discussion

3.1. Shapes studied

Different structures were modelled, including symmetrical and asymmetrical AEs. The symmetrical AE is obtained when $\mu_0 = \nu_0$ while asymmetrical dots correspond to any AE with parameters $\mu_0 \neq \nu_0$. In Fig. 2, three AE cross-sections are shown. The three AEs are of the same volume ($1.333\,\mathrm{m}^2 \times L$) with parameters (i) $\mu_0^2 = 1\,\mathrm{m}$, $\nu_0^2 = 1\,\mathrm{m}$; (ii) $\mu_0^2 = 1.3\,\mathrm{m}$, $\nu_0^2 = 0.7396\,\mathrm{m}$; and (iii) (i) $\mu_0^2 = 0.8\,\mathrm{m}$, $\nu_0^2 = 1.222\,\mathrm{m}$; corresponding to the upper, middle, and lower figures, respectively. For the symmetrical AE, the height and radius are the same. The medium in the enclosure is considered to be air with a sound speed equal to 343 m/s.

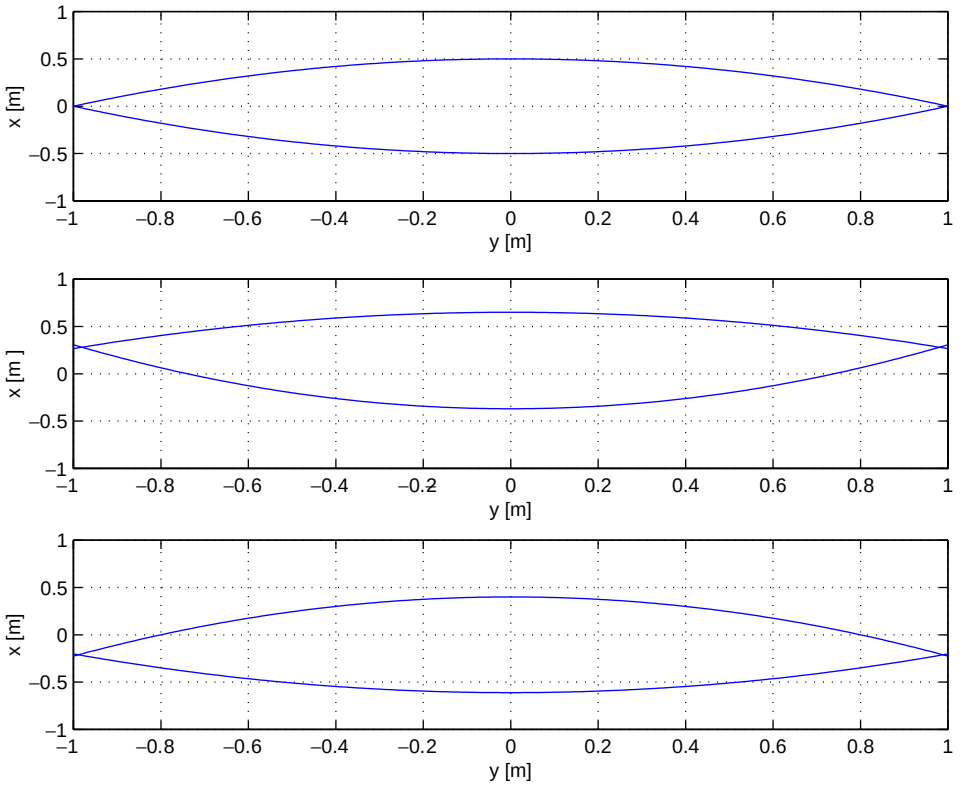

Fig. 2. Plot of three AEs having the same volume (cross-sectional view in z plane). The upper figure shows the AE with parameters: $\mu_0^2 = \nu_0^2 = 1\,\mathrm{m}$. The middle figure is for $\mu_0^2 = 1.3\,\mathrm{m}$ and the lower figure is for $\mu_0^2 = 0.8\,\mathrm{m}$.

3.2. Frequencies

In searching for frequencies, we started with α and scanned in β for zeroes of $dM_1/d\mu$ and dN_1/dv, $dM_2/d\mu$ and dN_2/dv, and dN_1/dv and dN_2/dv corresponding to case (a), (b), and (c) (refer to Section 2), respectively. This technique is illustrated in Fig. 3 for the symmetrical AE. In Fig. 3a, (α, β) values are plotted where $dM_1/d\mu$ and dN_1/dv have zeroes corresponding to case (a). Line codings for zeroes of $dM_1/d\mu$ and dN_1/dv are squares and diamonds, respectively. The simultaneous zeroes are the intersection points in a (α, β) plot; the solutions are even about $y = 0$. In a similar way, Fig. 3b shows the (α, β) values where $dM_2/d\mu$ and dN_2/dv are zero corresponding to case (b). Again, the simultaneous zeroes are the intersection points in a (α, β) plot; the solutions are odd about $y = 0$. In Fig. 3c, zeroes of dN_1/dv and dN_2/dv are plotted as a function of (α, β) corresponding to case (c). Evidently, no simultaneous zeroes are found (a close inspection of the curves will show that the dN_1/dv and dN_2/dv curves do not intersect). Thus, there will be no eigenstates of the type considered in case (c). This is to be expected on the basis of symmetry since the solutions must have definite parity about $y = 0$.

As mentioned earlier, it follows from Eqs. (10) and (11) that

$$M_1(\alpha, \beta; \gamma) = N_1(-\alpha, \beta; \gamma), \tag{45}$$

$$M_2(\alpha, \beta; \gamma) = N_2(-\alpha, \beta; \gamma) \tag{46}$$

for any value γ. Thus,

$$M_1(\alpha, \beta; \gamma)N_1(\alpha, \beta; \gamma) = N_1(-\alpha, \beta; \gamma)M_1(-\alpha, \beta; \gamma), \tag{47}$$

$$M_2(\alpha, \beta; \gamma)N_2(\alpha, \beta; \gamma) = N_2(-\alpha, \beta; \gamma)M_2(-\alpha, \beta; \gamma). \tag{48}$$

The latter relation applied to the case of a symmetric AE ($\gamma = \mu_0 = v_0$) shows that if (α, β) is a simultaneous zero point for $dM_1/d\mu$ and dN_1/dv (or $dM_2/d\mu$ and dN_2/dv) then $(-\alpha, \beta)$ is *also* a simultaneous zero point for $dM_1/d\mu$ and dN_1/dv (or $dM_2/d\mu$ and dN_2/dv). Applied to an asymmetric AE, it shows that the (μ_0, v_0) dot has the same frequency spectrum as the (v_0, μ_0), as expected from the mirror image shapes.

3.3. Eigenmodes and symmetry

Examples of eigenmodes for the symmetrical AE are given in Fig. 4. A trivial solution to the Helmholtz equation with Neumann boundary conditions always exists. This is the solution corresponding to case (a) with $\alpha = \beta = 0$. In this case, it follows from Eqs. (19) and (22) and the recursive relations (16) and (24) that $M_1(\mu)$ and $N_1(v)$ both are constant functions and so Neumann boundary conditions are trivially satisfied. Notice also that in this particular case (with $\alpha = \beta = 0$), it follows from Eqs. (43) and (42) that

$$f = p\frac{c}{2L} \quad \text{where } p = 0, 1, 2, 3, \ldots . \tag{49}$$

The first non-trivial solution (with smallest absolute value of β excluding the trivial solution) for the symmetrical AE occurs in case (b) with parameters $(\alpha, \beta) = (0, -4.4817)$ found by close inspection of the results shown in Fig. 3b. The degeneracy of this eigenfrequency is 1 and the corresponding eigenmode is shown in Fig. 4a. The second and third eigenmodes are degenerate

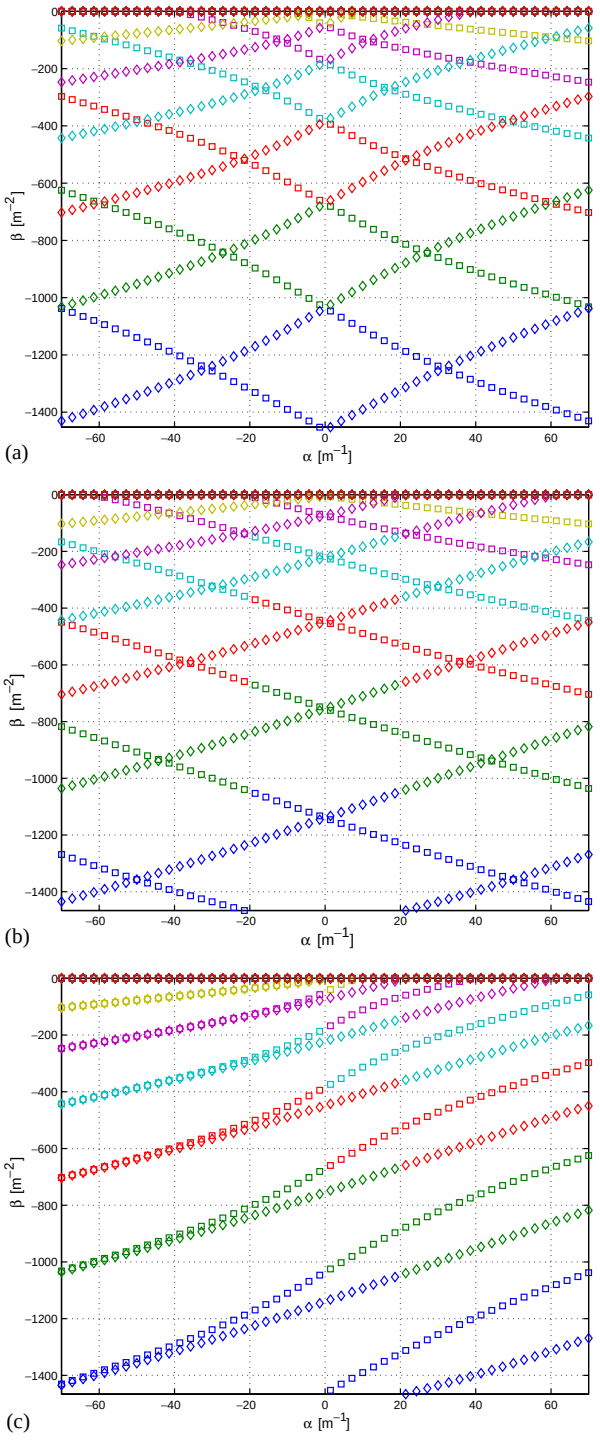

Fig. 3. Plot of the (α, β) values where $dM_1/d\mu$ and dN_1/dv (a), $dM_2/d\mu$ and dN_2/dv (b), and dN_1/dv and dN_2/dv (c), respectively, are zero for a symmetric AE.

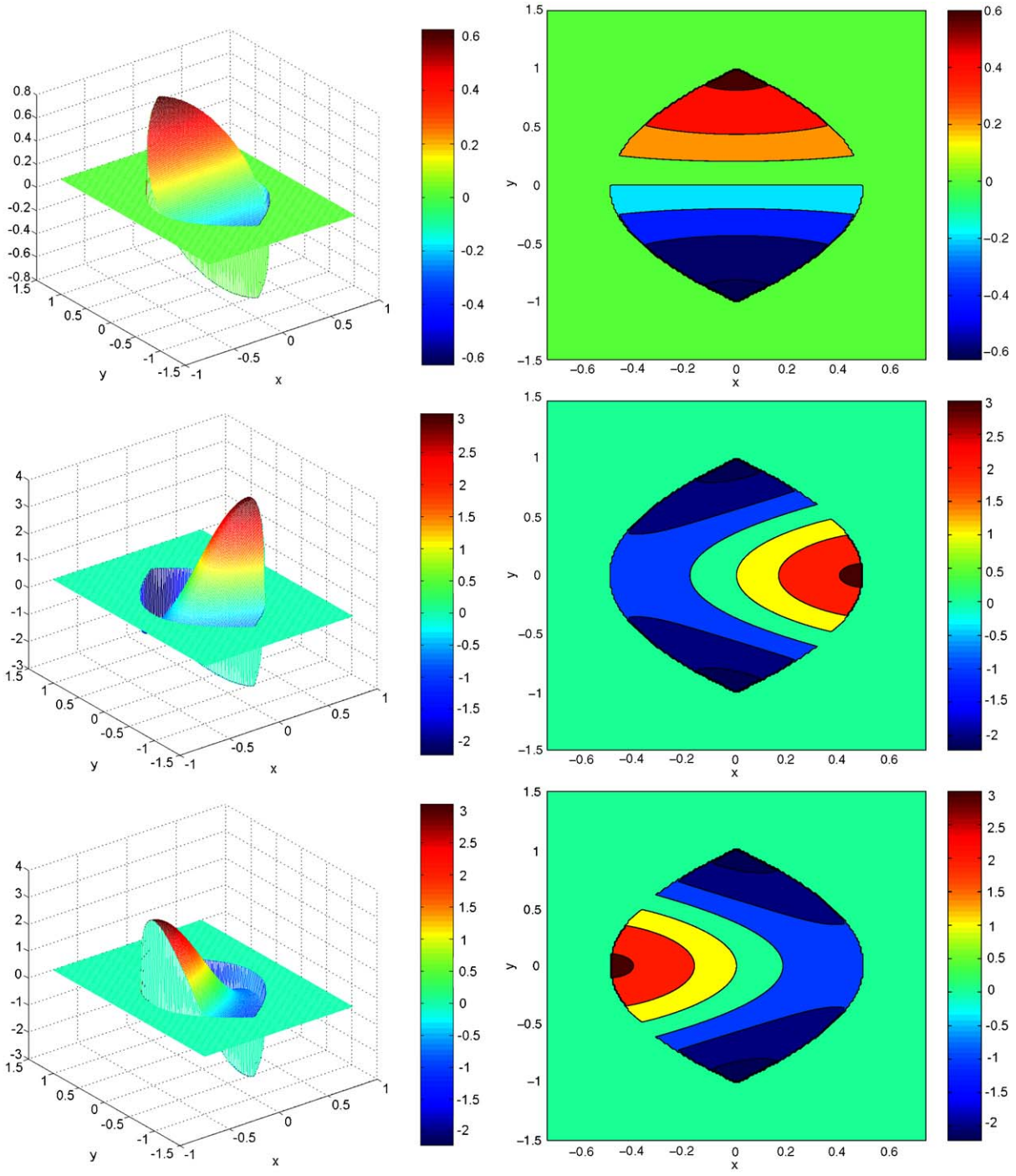

Fig. 4. Plots in three dimensions (left) and in contour (right) of the first three non-trivial eigenmodes for a symmetric AE. The upper plot shows the groundstate $(\alpha, \beta) = (0, -4.4817)$ (corresponding to case (b)), and the other two (middle and lower plots) are the second and third eigenmodes (both corresponding to case (a)) with parameters $(\alpha, \beta) = (\pm 6.037, -13.47)$.

states (i.e., they have the same β value) corresponding to case (a) with parameters $(\alpha,\beta) = (6.037,-13.47)$ and $(\alpha,\beta) = (-6.037,-13.47)$, respectively. The two eigenmodes are shown in Figs. 4b and c. Conclusions as to the symmetry properties of the eigenmodes can be deduced using the results in Section 2. For instance, states corresponding to case (a) where $B = D = 0$ is imposed satisfy

$$\begin{aligned}\psi(x,y,z) = \psi(\mu,v,z) &= M_1(\mu)N_1(v)Z(z) = M_1(\mu)N_1(-v)Z(z)\\ &= \psi(\mu,-v,z) = \psi(x,-y,z)\end{aligned} \tag{50}$$

and (if $\alpha = 0$),

$$\begin{aligned}\psi(x,y,z) = \psi(\mu,v,z) &= M_1(\mu)N_1(v)Z(z) = N_1(\mu)M_1(v)Z(z)\\ &= \psi(v,\mu,z) = \psi(-x,y,z),\end{aligned} \tag{51}$$

where use has been made of Eq. (45) in obtaining the third equality. Similarly, states corresponding to case (b) for which $A = C = 0$ are antisymmetric with respect to mirror reflections in the $y = 0$ plane, because:

$$\begin{aligned}\psi(x,y,z) = \psi(\mu,v,z) &= M_2(\mu)N_2(v)Z(z) = -M_2(\mu)N_2(-v)Z(z)\\ &= -\psi(\mu,-v,z) = -\psi(x,-y,z).\end{aligned} \tag{52}$$

In addition, if $\alpha = 0$, states corresponding to case (b) will be symmetric with respect to reflections in the $x = 0$ plane:

$$\psi(x,y,z) = \psi(-x,y,z), \tag{53}$$

following steps analogous to those used in deriving Eq. (51).

Thus, the first non-trivial solution is symmetric (antisymmetric) with respect to a mirror reflection in the $x = 0$ ($y = 0$) plane since $(\alpha,\beta) = (0,-4.4817)$ as this state belongs to case (b) in agreement with Fig. 4a. The second and third non-trivial solutions have α values different from zero (and belong to case (a)). These solutions are therefore symmetric with respect to a mirror reflection in the $y = 0$ plane but they show no symmetry with respect to a mirror reflection in the $x = 0$ plane. This is in agreement with Figs. 4b and c.

3.4. Shape dependence of frequencies

We next consider the frequency dependence on AE asymmetry—as specified by μ_0—for a given volume equal to the volume of a symmetrical AE with parameters $\mu_0^2 = v_0^2 = 1\,\mathrm{m}$ (Fig. 5). It is evident that the groundstate of the symmetrical AE has a lower eigenfrequency as compared to the asymmetrical AEs although the variation in eigenfrequency is small (ranging from 115.6 Hz when $\mu_0^2 = 1\,\mathrm{m}$ to 116.6 Hz (117.4 Hz) when $\mu_0^2 = 0.8\,\mathrm{m}$ (1.3 m)). The shape dependence is quite different from, for example, a parabolic rotational lens, where the latter has a local maximum for the symmetrical structure surrounded by two minima [10]. We also note that in the case of a circular cylinder enclosure of the same volume, the groundstate frequency is

$$f_{\mathrm{cyl}} = \frac{cj_{11}}{2\pi r_c}, \tag{54}$$

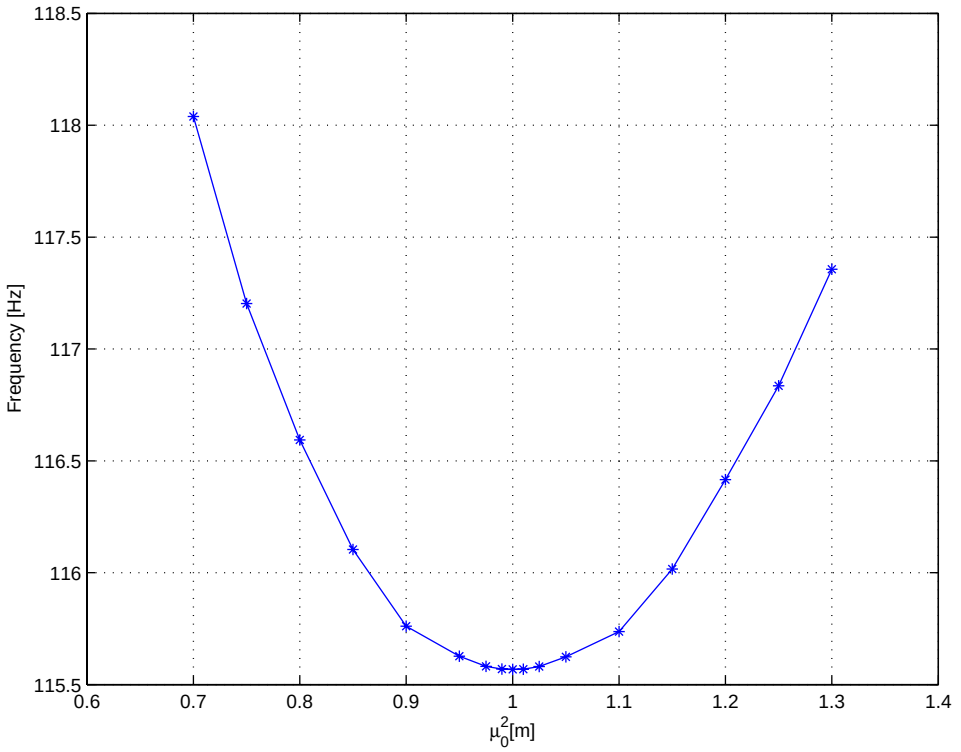

Fig. 5. Plot of the groundstate frequency for a series of AEs having the same volume. Parameters for the symmetrical AE are: $\mu_0^2 = v_0^2 = 1\,\mathrm{m}$.

where j_{11} is the smallest non-zero root (equal to 1.841) of the Bessel function derivative: J'_m and r_c is the radius of a circle having the same cross-sectional area as the PCC enclosure considered in Fig. 5, i.e., $r_c = 0.6515\,\mathrm{m}$. Inserting these values in Eq. (54) gives

$$f_{\mathrm{cyl}} = 154\,\mathrm{Hz}, \tag{55}$$

i.e., the minimum (non-trivial) frequency for a circular cylinder rigid enclosure is higher than the minimum (non-trivial) frequency for a parabolic cylinder enclosure of the same volume. This is expected due to the larger degree of asymmetry in the latter (PCC) enclosure.

4. Conclusions

An exact method for finding eigenmodes and eigenvalues in an acoustic enclosure with rigid walls defined by two confocal parabolic cylinders and two plane surfaces using parabolic cylinder coordinates is presented. Eigenmode symmetry properties are discussed along with the fundamental-mode frequency dependence on AE asymmetry for a series of AEs having the same volume. It is concluded that the symmetrical AE has the lowest groundstate frequency.

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